

# Nash modification on toric surfaces

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## Abstract

It has been recently shown that the iteration of Nash modification on not necessarily normal toric varieties corresponds to a purely combinatorial algorithm on the generators of the semigroup associated to the toric variety. We will show that for toric surfaces this algorithm stops for certain choices of affine charts of the Nash modification. In addition, we give a bound on the number of steps required for the algorithm to stop in the cases we consider. Let  $\mathbb{C}(x_1, x_2)$  be the field of rational functions of a toric surface. Then our result implies that if  $\nu : \mathbb{C}(x_1, x_2) \rightarrow \Gamma$  is any valuation centered on the toric surface and such that  $\nu(x_1) \neq \lambda \nu(x_2)$  for all  $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ , then a finite iteration of Nash modification gives local uniformization along  $\nu$ .

## Introduction

We are interested in applying the Nash modification to not necessarily normal toric surfaces and finding out whether or not the iteration of this process resolves their singularities. The Nash modification of an equidimensional algebraic variety replaces singular points by limits of tangent spaces to non-singular points. Following the work of Nobile ([8]), Rebassoo showed in his thesis ([9]) that the iteration of Nash modification resolves the singularities of the family  $\{z^p + x^q y^r = 0\} \subset \mathbb{C}^3$ , for any positive integers  $p, q, r$  without a common divisor. In the context of normal toric varieties (over an algebraically closed field of characteristic zero), Gonzalez-Sprinberg ([3]), and later Lejeune-Jalabert and Reguera ([7]), have exhibited an ideal, called the log-jacobian ideal, whose blowing-up is the Nash modification of the toric variety. Using the work of Gonzalez-Sprinberg ([3], [4]), and Hironaka ([6]), Spivakovsky ([10]) proved that iterating Nash modification composed with normalization resolves singularities of surfaces. More recently, normalized Nash modification has appeared in the work of Atanasov et al. ([1]). Moreover, it has been recently shown by González Perez and Teissier ([2]), and by Grigoriev and Milman ([5]), that for the case of (not necessarily normal) toric varieties of any dimension, the iteration of Nash modification can be translated into a purely combinatorial algorithm.

Here we follow the results proved in [5]. Let  $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$  be a set of monomial exponents of some toric surface  $X$ , i. e.,  $X$  is the Zariski closure in  $\mathbb{C}^r$  of  $\{(x^{\gamma_1}, \dots, x^{\gamma_r}) | x \in (\mathbb{C}^*)^2\}$ , where  $x^{\gamma_i} = x_1^{\gamma_{i,1}} \cdot x_2^{\gamma_{i,2}}$ . Let  $S = \{\{i, j\} \subset \{1, \dots, r\} | \det(\gamma_i \ \gamma_j) \neq 0\}$ . Fix  $\{i_0, j_0\} \in S$  and let

$$\begin{aligned} A_{i_0}(\xi) &= \{\gamma_k - \gamma_{i_0} | k \in \{1, \dots, r\} \setminus \{i_0, j_0\}, \det(\gamma_k \ \gamma_{j_0}) \neq 0\}, \\ A_{j_0}(\xi) &= \{\gamma_k - \gamma_{j_0} | k \in \{1, \dots, r\} \setminus \{i_0, j_0\}, \det(\gamma_k \ \gamma_{i_0}) \neq 0\}. \end{aligned}$$

Let  $\xi_{i_0, j_0} = A_{i_0}(\xi) \cup A_{j_0}(\xi) \cup \{\gamma_{i_0}, \gamma_{j_0}\}$  and  $S' = \{\{i, j\} \in S \mid (0, 0) \notin \text{Conv}(\xi_{i, j})\}$ , where  $\text{Conv}(\xi_{i, j})$  denotes the convex hull of  $\xi_{i, j}$  in  $\mathbb{R}^2$ . Then it is proved in [5] (section 4) that, if  $(0, 0) \notin \text{Conv}(\xi)$ , the affine charts of Nash modification of  $X$  are given by the toric surfaces associated to the sets  $\xi_{i, j}$  such that  $\{i, j\} \in S'$ . The iteration of this algorithm gives rise to a tree in which every branch corresponds to the choices of  $\{i, j\} \in S'$ . A branch of the algorithm ends if the semigroup  $\mathbb{Z}_{\geq 0}\xi_{i, j}$  is generated by two elements.

We will prove the following result: Fix  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x_1, x_2) \mapsto ax_1 + bx_2$ , where  $a, b \in \mathbb{Z}$  and  $(a, b) = 1$  (we allow  $a = 1, b = 0$ , and  $a = 0, b = 1$ ), such that  $L(\xi) \geq 0$ . Let  $\gamma_i, \gamma_j \in \xi$  be two elements such that  $L(\gamma_i) \leq L(\gamma_k)$  for all  $\gamma_k \in \xi$ ,  $L(\gamma_j) \leq L(\gamma_k)$  for all  $\gamma_k \in \xi$  such that  $\det(\gamma_i, \gamma_k) \neq 0$  and such that  $\{i, j\} \in S'$ . We say that  $L$  chooses  $\gamma_i, \gamma_j$ , even though, as we will see later,  $\gamma_i, \gamma_j$  need not be uniquely determined by the above conditions. We will prove that the iteration of the algorithm stops for all the possible choices of  $L$ . In addition, we give a bound (that depends on  $L$ ) on the number of steps required for the algorithm to stop. Of course, this result gives only some progress towards the question of whether or not Nash modification resolves singularities of toric surfaces. If we could prove that the previous statement is true for an arbitrary linear transformation  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $L(\xi) \geq 0$  then the problem would be solved.

Our result has the following interpretation in terms of valuations. Let  $X$  be the affine toric surface determined by  $\xi \subset \mathbb{Z}^2$  and let  $\mathbb{C}(x_1, x_2)$  be its field of rational functions. Fix some linear transformation  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x_1, x_2) \mapsto ax_1 + bx_2$  as before, i. e.,  $a, b \in \mathbb{Z}$  relatively prime and such that  $L(\xi) \geq 0$ . Consider any valuation  $\nu : \mathbb{C}(x_1, x_2) \rightarrow \mathbb{R}$  such that  $\nu(x_1) = a, \nu(x_2) = b$ . Since  $L(\xi) \geq 0$  such a valuation is centered at a point of  $X$ . Suppose that  $L$  chooses the affine chart given by  $\gamma_{i_0}, \gamma_{j_0}$ . Since once again  $L(\xi_{i_0, j_0}) \geq 0$  we have that  $\nu$  has a center on this affine chart. Continuing this way, the center of the valuation at every application of Nash modification describes the same branch (or branches) as  $L$ . Conversely, if  $\nu : \mathbb{C}(x_1, x_2) \rightarrow \mathbb{R}$  is any valuation centered at a point of  $X$  such that  $\frac{\nu(x_1)}{\nu(x_2)} \in \mathbb{Q}$ , we can recover the linear function  $L$  as above such that the center of  $\nu$  after successive Nash modifications belongs to one of the charts chosen by  $L$ .

Fix again some  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x_1, x_2) \mapsto ax_1 + bx_2$  as before and now consider any valuation  $\nu : \mathbb{C}(x_1, x_2) \rightarrow \mathbb{Z}_{lex}^2$  centered on  $X$  and such that  $\nu(x_1) = (a, c), \nu(x_2) = (b, d)$  where  $(a, b) \neq q(c, d)$  for all  $q \in \mathbb{Q}$ . In this case, whenever  $L$  has a unique choice of an affine chart, then the center of  $\nu$  is in this affine chart. If  $L$  has two possible choices (we will see later that if the choice of affine chart of  $L$  is not unique, then there are at most two possible choices) then the center of  $\nu$  will be in one and only one of these affine charts. Continuing this way, we obtain once again that the center of the valuation will describe one possible branch of  $L$ .

By the result, the branches determined by  $L$  are finite and they end in a non-singular surface. In particular, the center of the valuations considered before is

non-singular, that is, this process gives us local uniformization along  $\nu$ . Notice that we can replace  $L$  by any linear transformation  $L'$  such that  $\ker L = \ker L'$  since both  $L$  and  $L'$  will choose the same elements at each step of the algorithm. All this implies the following theorem:

**Theorem 0.1.** *Let  $\nu : \mathbb{C}(x_1, x_2) \rightarrow \Gamma$  be any valuation centered on  $X$  such that  $\nu(x_1) \neq \lambda\nu(x_2)$  for all  $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ . Then a finite iteration of Nash modification gives local uniformization along  $\nu$ .*

In other words, the problem of local uniformization of toric surfaces by iterating Nash modification remains open only for the following set of valuations: the valuations  $\nu$  of real rank 1 and rational rank 2, such that there exists  $\lambda \in \mathbb{R} \setminus \mathbb{Q}$  such that  $\nu(x_1) = \lambda\nu(x_2)$ . I would like to mention that I have been informed that Pedro González Perez and Bernard Teissier have obtained a similar result for toric varieties of any dimension.

Finally, I would like to express my sincere gratitude to Mark Spivakovsky, whose constant support and guidance have been of great help to obtain the results presented here. Among other things, he guided me through the interpretation of the result in terms of valuations.

## 1 The algorithm

In this section we give an explicit description of the algorithm as stated in [5] and of the concrete affine charts of the Nash modification of a toric surface that we will follow.

**Definition 1.1.** *Let  $X \subset \mathbb{C}^r$  be an algebraic variety of pure dimension  $m$ . Consider the Gauss map:*

$$\begin{aligned} G : X \setminus \text{Sing}(X) &\rightarrow G(m, r) \\ x &\mapsto T_x X, \end{aligned}$$

where  $G(m, r)$  is the Grassmanian parameterizing the  $m$ -dimensional vector spaces in  $\mathbb{C}^r$ . Denote by  $X^*$  the Zariski closure of the graph of  $G$ . Call  $\nu$  the restriction to  $X^*$  of the projection of  $X \times G(m, r)$  to  $X$ . The pair  $(X^*, \nu)$  is called the Nash modification of  $X$ .

We next define our main object of study.

**Definition 1.2.** *Let  $\xi := \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$  such that  $\mathbb{Z}\xi := \{\sum_{k=1}^r \lambda_k \gamma_k \mid \lambda_k \in \mathbb{Z}\} = \mathbb{Z}^2$ . Consider the following monomial map:*

$$\begin{aligned} \Phi : (\mathbb{C}^*)^2 &\rightarrow \mathbb{C}^r \\ x = (x_1, x_2) &\mapsto (x^{\gamma_1}, \dots, x^{\gamma_r}), \end{aligned}$$

where  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ,  $x^{\gamma_k} := x_1^{\gamma_{k,1}} \cdot x_2^{\gamma_{k,2}}$  for  $k = 1, \dots, r$  and  $\gamma_k = (\gamma_{k,1}, \gamma_{k,2})$ . Let  $X$  denote the Zariski closure of the image of  $\Phi$ . We call  $X$  an affine toric variety and  $\xi$  a set of monomial exponents of  $X$ .

It is known that  $X$  is an irreducible surface that contains an algebraic group isomorphic to  $(\mathbb{C}^*)^2$  that extends to  $X$  the natural action on itself (see [11], Chapter 13).

The following is a step-by-step description of the Nash modification algorithm for toric surfaces as proved in [5] (Section 4):

- (A1) Let  $\xi := \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$  be a set of monomial exponents of some toric surface  $X$  such that  $(0, 0) \notin \text{Conv}(\xi)$ .
- (A2) Let  $S := \{\{i, j\} \subset \{1, \dots, r\} \mid \det(\gamma_i, \gamma_j) \neq 0\}$ . Fix some  $\{i_0, j_0\} \in S$  and consider the sets

$$\begin{aligned} A_{i_0}(\xi) &:= \{\gamma_k - \gamma_{i_0} \mid k \in \{1, \dots, r\} \setminus \{i_0, j_0\}, \det(\gamma_k, \gamma_{j_0}) \neq 0\}, \\ A_{j_0}(\xi) &:= \{\gamma_k - \gamma_{j_0} \mid k \in \{1, \dots, r\} \setminus \{i_0, j_0\}, \det(\gamma_k, \gamma_{i_0}) \neq 0\}. \end{aligned}$$

- (A3) Consider  $\xi_{i_0, j_0} := A_{i_0}(\xi) \cup A_{j_0}(\xi) \cup \{\gamma_{i_0}, \gamma_{j_0}\}$ . If  $(0, 0) \notin \text{Conv}(\xi_{i_0, j_0})$ , then this set is a set of monomial exponents for one affine chart of the Nash modification of  $X$ .

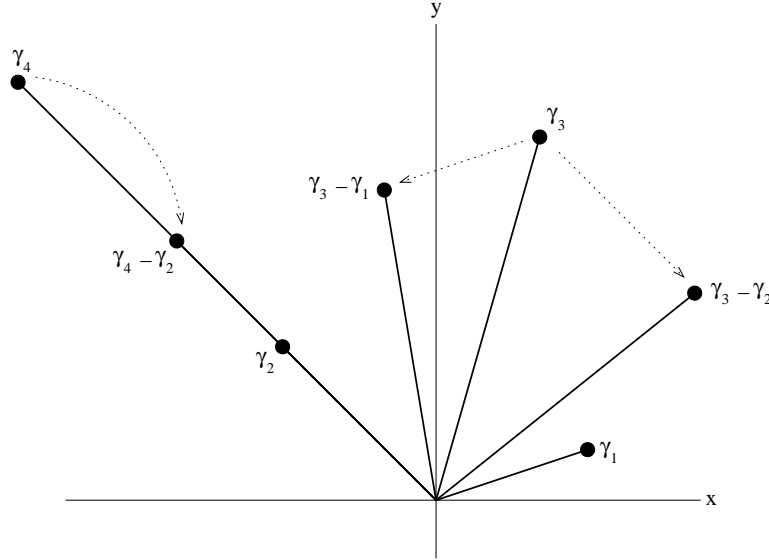


Figure 1: Step (A3) of the algorithm for  $\{1, 2\} \in S$ .

- (A4) If the semigroup  $\mathbb{Z}_{\geq 0}\xi_{i_0, j_0}$  is generated by two elements then this affine chart is non-singular and we stop. Otherwise, replace  $\xi$  by  $\xi_{i_0, j_0}$  and repeat the process.

**Remark 1.3.** Notice that we can choose any set of generators  $\xi'$  of the semigroup  $\mathbb{Z}_{\geq 0}\xi_{i_0, j_0}$  since the resulting toric surfaces will be isomorphic. Moreover, if  $\mathbb{Z}_{\geq 0}\xi' = \mathbb{Z}_{\geq 0}\xi_{i_0, j_0}$  then it is also clear that  $\mathbb{Z}\xi' = \mathbb{Z}^2$ . We say that  $\xi' \subset \mathbb{Z}_{\geq 0}\xi$  is a minimal set of monomial exponents if  $\xi'$  generates  $\mathbb{Z}_{\geq 0}\xi$  as a semigroup and for all  $\gamma \in \xi'$ ,  $\gamma \notin \mathbb{Z}_{\geq 0}(\xi' \setminus \{\gamma\})$ .

In this paper, we will only consider the elements of  $S$  obtained in the following way:

- (B1) Fix any linear transformation  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x_1, x_2) \mapsto ax_1 + bx_2$ ,  $a, b \in \mathbb{Z}$ , and  $(a, b) = 1$  (we allow  $a = 1, b = 0$ , and  $a = 0, b = 1$ ), such that  $L(\xi) \geq 0$ . We call  $L(\gamma)$  the  $L$ -value of  $\gamma$ .
- (B2) Let  $\gamma_i, \gamma_j \in \xi$  be two elements such that  $\{i, j\} \in S$ ,  $L(\gamma_i) \leq L(\gamma_k)$  for all  $\gamma_k \in \xi$ ,  $L(\gamma_j) \leq L(\gamma_k)$  for all  $\gamma_k \in \xi$  such that  $\det(\gamma_i \ \gamma_k) \neq 0$ , and such that  $(0, 0) \notin \text{Conv}(\xi_{i, j})$ . We say that  $L$  chooses  $\gamma_i$  and  $\gamma_j$ .

**Remark 1.4.** As we will see later, the choices of  $L$  in (B2) may not be unique (cf. lemma 3.1).

**Example 1.5.** Let  $\gamma_1 = (1, 0), \gamma_2 = (2, 1), \gamma_3 = (0, 2), \gamma_4 = (0, 3)$ .

- (A1) Let  $\xi = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \subset \mathbb{Z}^2$ . Then  $S = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$ .

- (B1) Consider the following linear transformations:

- (i)  $L_1(x, y) = y$ .
- (ii)  $L_2(x, y) = x + y$ .

- (B2) (i)  $L_1$  chooses  $\gamma_1$  and  $\gamma_2$ .  
(ii)  $L_2$  chooses  $\gamma_1$  and  $\gamma_3$ .

- (A2) For the choices  $\{1, 2\}, \{1, 3\}$  we obtain, respectively:

- (i)  $A_1(\xi) = \{\gamma_3 - \gamma_1, \gamma_4 - \gamma_1\}$ ,  $A_2(\xi) = \{\gamma_3 - \gamma_2, \gamma_4 - \gamma_2\}$ .
- (ii)  $A_1(\xi) = \{\gamma_2 - \gamma_1\}$ ,  $A_3(\xi) = \{\gamma_2 - \gamma_3, \gamma_4 - \gamma_3\}$ .

- (A3) The resulting sets are, respectively:

- (i)  $\xi_{1,2} = \{(-1, 2), (-1, 3)\} \cup \{(-2, 1), (-2, 2)\} \cup \{(1, 0), (2, 1)\}$ .
- (ii)  $\xi_{1,3} = \{(1, 1)\} \cup \{(2, -1), (0, 1)\} \cup \{(1, 0), (0, 2)\}$ .

(A4) The semigroups  $\mathbb{Z}_{\geq 0}\xi_{1,2}$ ,  $\mathbb{Z}_{\geq 0}\xi_{1,3}$  are generated by, respectively:

- (i)  $\{(-2, 1), (1, 0)\}$ . Therefore the algorithm stops for  $L_1$ .
- (ii)  $\{(0, 1), (1, 0), (2, -1)\}$ . Replacing  $\xi$  by  $\xi_{1,3}$ , we have to repeat the process for  $L_2$ .

What we intend to prove is that the algorithm stops for any choice of linear transformation such that its kernel has rational slope or infinite slope. In other words, we will show that in this case it is always possible to obtain a semigroup generated by two elements after iterating the algorithm enough times.

## 2 A first case

In this section we study a first case of the problem stated in the previous section. Consider a set of monomial exponents given by  $\xi = \{(1, 0), \gamma_1, \dots, \gamma_r\} \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ . We will iterate the algorithm following the choices of the linear transformation  $L(x, y) = y$  and we show that one eventually arrives to a semigroup generated by two elements (actually, those elements will be  $(1, 0)$  and  $(\lambda, 1)$  for some  $\lambda \in \mathbb{Z}$ ).

We intend to prove (always by following  $L(x, y) = y$ ):

- (1) If  $\xi = \{(1, 0), (a_1, b_1), \dots, (a_r, b_r)\} \subset \mathbb{Z}^2$  is such that

- (i)  $\mathbb{Z}\xi = \mathbb{Z}^2$ ,
- (ii)  $b_i > 1$  for all  $i$ ,

then by iterating the algorithm we eventually arrive to an element of the form  $(\lambda, 1)$  which can be taken by a linear isomorphism (that preserves  $L$ ) to  $(0, 1)$ .

- (2) If  $\xi = \{(1, 0), (0, 1), (-a_1, b_1), \dots, (-a_r, b_r)\}$  is a minimal set of monomial exponents of some toric surface where (necessarily, possibly after renumbering)  $1 \leq a_1 < a_2 < \dots < a_r$  and  $1 < b_1 < b_2 < \dots < b_r$ , then by iterating the algorithm one eventually arrives to a semigroup generated by two elements.

Therefore, (1) implies that whenever  $(1, 0) \in \xi$  we can also suppose that  $(0, 1) \in \xi$ , i. e., the situation in (2). The isomorphism that we will apply in (1) will be an element of  $SL(2, \mathbb{Z})$  that preserves  $L$ . Notice that we can do this without modifying the algorithm since such an isomorphism induces an automorphism of the field of rational functions of the toric surface associated to  $\xi$ .

**Lemma 2.1.** *For  $\xi = \{(1, 0), (a_1, b_1), (a_2, b_2), \dots, (a_r, b_r)\}$  as in (1), the iteration of the algorithm eventually produces an element of the form  $(\lambda, 1)$ .*

*Proof.* Since  $\mathbb{Z}\xi = \mathbb{Z}^2$  we have  $\gcd(b_1, b_2, \dots, b_r) = 1$  and we assume that  $1 < b_1 < b_2 < \dots < b_r$ . We can assume this since if there were two points with the same  $L$ -value then one of them would be generated by the other and some multiple of  $(1, 0)$ . In addition, this property remains true after applying the algorithm because the first choice of  $L$  is always  $(1, 0)$ .

Call  $\gamma_0 = (1, 0)$  and  $\gamma_i = (a_i, b_i)$ . Then  $L$  chooses  $\gamma_0$  and  $\gamma_1$  and applying once the algorithm we replace  $\xi$  by  $\xi_{0,1}$ . As before, we consider a subset of  $\xi_{0,1}$  that includes only one element for every possible  $L$ -value (see figure 2). Of course, this is a set of generators of  $\mathbb{Z}_{\geq 0}\xi_{0,1}$ . Then we repeat the process taking into account this consideration. Having this in mind, now it suffices to study the effect of the algorithm on the second coordinate.

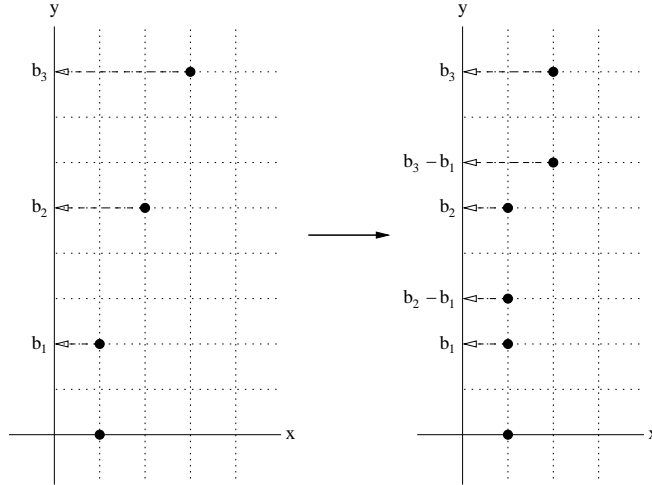


Figure 2: One element for every possible  $L$ -value.

We begin with integers  $1 < b_1 < b_2 < \dots < b_r$  such that  $\gcd(b_1, b_2, \dots, b_r) = 1$ . After applying once the algorithm we obtain a new set  $\xi'$  such that the set of its elements' second coordinates contains the subset  $\{b_1, b_2 - b_1, \dots, b_r - b_1\}$ . Since  $\gcd(b_1, b_2 - b_1, \dots, b_r - b_1) = 1$  we still have that the greatest common divisor of the second coordinate of all points in  $\xi'$  is 1. We repeat the algorithm until we find some  $n_1 \in \mathbb{N}$  such that  $b_2 - n_1 b_1 \leq b_1$  and  $b_2 - (n_1 - 1)b_1 > b_1$ .

If  $b_2 - n_1 b_1 = b_1$  then  $b_2$  is a multiple of  $b_1$  and nothing happens. We keep repeating the algorithm until we find some  $n_2 \in \mathbb{N}$  such that  $b_3 - n_2 b_1 \leq b_1$  and  $b_3 - (n_2 - 1)b_1 > b_1$ . Again, if  $b_3 - n_2 b_1 = b_1$  then  $b_3$  is a multiple of  $b_1$ . This situation cannot continue for all  $b_i$  since  $\gcd(b_1, b_2, \dots, b_r) = 1$ . Therefore,  $b_i - n b_1 < b_1$  for some  $2 \leq i \leq r$  and some  $n \in \mathbb{N}$ . At this moment, we have a new set  $\xi'$  with

some element whose second coordinate is smaller than  $b_1$  and such that the greatest common divisor of the second coordinate of all its elements is 1, that is, we are in the same situation we began with.

Since all numbers involved are integers, this process will take us eventually to 1, that is, we will obtain an element of the form  $(\lambda, 1)$ , with  $\lambda \in \mathbb{Z}$ . Applying the linear isomorphism  $T(x, y) = (x - \lambda y, y)$  we finally have  $T(\lambda, 1) = (0, 1)$  and  $T(1, 0) = (1, 0)$ .  $\square$

Notice that when  $r = 2$  in the previous lemma the result of the algorithm on the second coordinate is precisely Euclid's algorithm for  $b_1$  and  $b_2$ . This observation directly implies the lemma in this case. We now proceed to prove (2).

**Lemma 2.2.** *Let  $\xi = \{(1, 0), (0, 1), (-a, b)\}$  where  $a \geq 1$  and  $b > 1$ . Then the iteration of the algorithm eventually produces a semigroup generated by two elements.*

*Proof.* We prove by induction that after applying the algorithm  $n$  times where  $n < b$  one obtains:

$$\{\delta_{n,i} | i = 0, 1, \dots, n\} \cup \{e_1, e_2\},$$

where  $\delta_{n,i} := (-a - (n-i), b-i)$ ,  $e_1 = (1, 0)$ , and  $e_2 = (0, 1)$  (see figure 3). Let  $n = 1$ . Since  $b > 1$ ,  $L$  chooses  $e_1$  and  $e_2$ . Then the algorithm gives  $\{(-a-1, b), (-a, b-1)\} \cup \{e_1, e_2\}$ , which is precisely  $\{\delta_{1,i} | i = 0, 1\} \cup \{e_1, e_2\}$ .

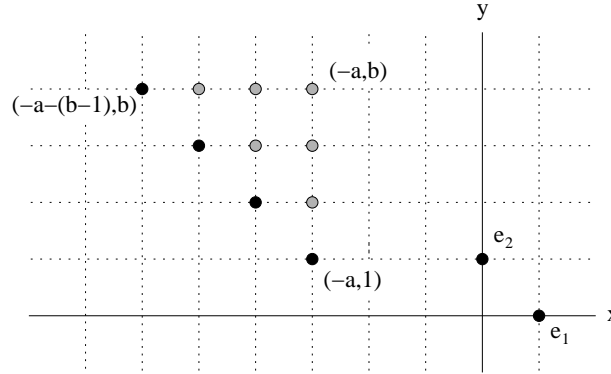


Figure 3: The resulting set.

Suppose that the statement is true for  $n - 1$ . So, after applying the algorithm  $n - 1$  times, we obtain:

$$\{\delta_{n-1,i} | i = 0, 1, \dots, n - 1\} \cup \{e_1, e_2\}.$$

Since  $n - 1 < b$ ,  $L$  chooses again  $e_1$  and  $e_2$ . Apply the algorithm again. Since  $\det(\delta_{n-1,i} \ e_1) \neq 0$  and  $\det(\delta_{n-1,i} \ e_2) \neq 0$  one takes  $\{\delta_{n-1,i} - e_1 | i = 0, 1, \dots, n - 1\}$



and  $\{\delta_{n-1,i} - e_2 | i = 0, 1, \dots, n-1\}$ . But  $\delta_{n-1,i} - e_1 = \delta_{n,i}$  and  $\delta_{n-1,i} - e_2 = \delta_{n,i+1}$ , which completes the induction. In particular, for  $n = b-1$  we obtain the set:

$$\xi' = \{(-a - (b - 1), b), (-a - (b - 2), b - 1), \dots, (-a, 1)\} \cup \{e_1, e_2\}.$$

Notice that the points  $(-a - (n - i), b - i)$  for  $i = 0, 1, \dots, n$  are all contained in some line  $l_n$  of slope -1, for each  $n$ . Now, since  $-\frac{1}{a} \geq -1$ , this implies, for  $n = b - 1$ , that every point in  $\xi'$  is generated by  $(-a, 1)$  and  $(1, 0)$ . Therefore, after  $b - 1$  steps, the resulting semigroup is generated by two elements.  $\square$

**Proposition 2.3.** *Let  $\xi = \{(1, 0), (0, 1), (-a_1, b_1), \dots, (-a_r, b_r)\}$ , where  $1 \leq a_1 < a_2 < \dots < a_r$  and  $1 < b_1 < \dots < b_r$ , be as in (2). Then the iteration of the algorithm eventually produces a semigroup generated by two elements.*

*Proof.* We proceed by induction on the number of elements of  $\xi$ . The case  $r = 1$  is given by the previous lemma. Assume that the result holds for  $r - 1$ . As in the previous lemma, after applying the algorithm  $b_1 - 1$  times every  $(-a_j, b_j)$  gives rise to (see figure 4):

$$\xi'_j := \{(-a_j - (b_1 - 1 - i), b_j - i) | i = 0, 1, \dots, b_1 - 1\}.$$

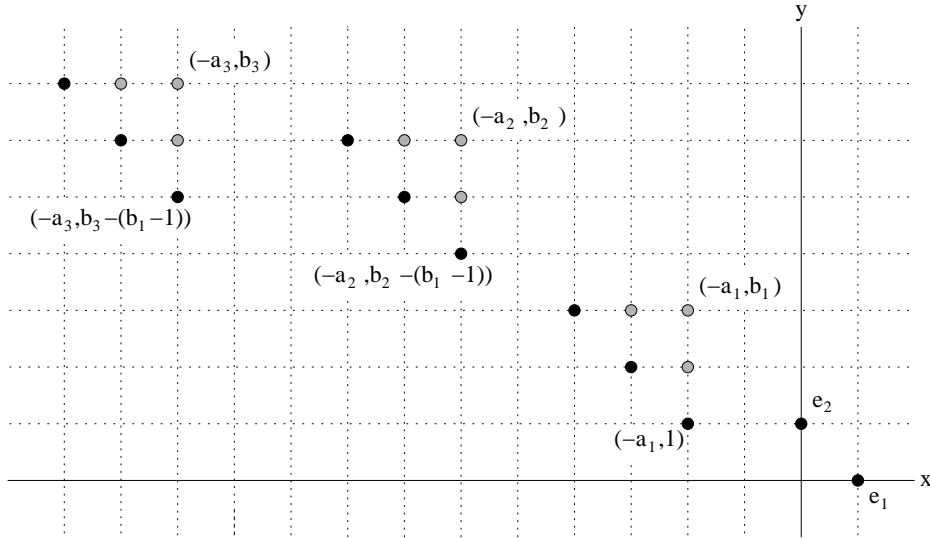


Figure 4: The resulting sets.

As before, each  $\xi_j'$  is contained in some line of slope -1. Therefore, since  $-\frac{1}{a_1} \geq -1$ , every element in  $\xi_j'$  is generated by  $(-a_j, b_j - (b_1 - 1))$ ,  $(-a_1, 1)$ , and  $(1, 0)$  for

each  $j$ . Therefore, if  $\xi' = \{(-a_i, b_i - (b_1 - 1)) | i = 2, \dots, r\} \cup \{(-a_1, 1), (1, 0)\}$ , we have

$$\bigcup_{j=1}^r \xi'_j \cup \{(1, 0), (0, 1)\} \subset \mathbb{Z}_{\geq 0} \xi'.$$

Next, we consider the linear isomorphism  $T(x, y) = (x + a_1 y, y)$ . Then we have (since  $T(-a_1, 1) = (0, 1)$ ),

$$T(\xi') = \{(1, 0), (0, 1), (-c_2, d_2), (-c_3, d_3), \dots, (-c_r, d_r)\}.$$

Since  $|\xi| = r + 2$  and  $|T(\xi')| = r + 1$  we have, by induction, that the iteration of the algorithm over  $\xi$  eventually produces a semigroup generated by two elements.  $\square$

**Remark 2.4.** Notice that if  $\xi = \{(-1, 0), (a_1, b_1), \dots, (a_r, b_r)\}$  then analogous results (1) and (2) for this set can be reduced to the previous ones by considering the linear isomorphism  $T(x, y) = (-x, y)$ , since this isomorphism preserves  $L$ .

### 3 Rational chart

Consider any set of monomial exponents given by  $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$ . In this section we are going to prove that the iteration of the algorithm following  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto (ax + by)$  where  $a, b \in \mathbb{Z}$  (which can be assumed to be relatively prime) and such that  $L(\xi) \geq 0$ , eventually produces a semigroup generated by two elements. To reach this goal, we intend to reduce this case to the one already solved. Under these assumptions we can assume that  $\xi \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$  and that  $L(x, y) = y$  (it suffices to take the isomorphism  $T(x, y) = (\beta x - \alpha y, ax + by)$ , where  $\alpha a + \beta b = 1$ ).

We intend to prove (always by following  $L(x, y) = y$ ):

- (1) If  $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$  such that  $L(\gamma_i) > 0$  for all  $i$ , then by iterating the algorithm we eventually arrive to an element of the form  $(n, 0)$ , with  $n \in \mathbb{Z}$ .
- (2) If  $\xi = \{(n, 0), \gamma_1, \dots, \gamma_r\}$  is a set of monomial exponents of some toric surface with  $n > 0$ , then the iteration of the algorithm eventually produces the point  $(1, 0)$ .

**Lemma 3.1.** *If  $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$  such that  $L(\gamma_i) > 0$  for all  $i$ , then by iterating the algorithm we eventually arrive to an element of the form  $(n, 0)$ , with  $n \in \mathbb{Z}$ .*

*Proof.* First, notice that the choices of  $L$  are not unique in the following cases (see figure 5):

- (i) There exist at least three elements  $\gamma_1, \gamma_2, \gamma_3$  such that

$$0 < L(\gamma_1) = L(\gamma_2) = L(\gamma_3) \leq L(\gamma'),$$

for all  $\gamma' \in \xi \setminus \{\gamma_1, \gamma_2, \gamma_3\}$ .

- (ii) There exists  $\gamma \in \xi$  such that  $0 < L(\gamma) < L(\gamma')$  for all  $\gamma' \in \xi \setminus \{\gamma\}$  and there are at least two elements  $\gamma_1, \gamma_2$ , with both  $\det(\gamma \gamma_i) \neq 0$  and such that

$$0 < L(\gamma) < L(\gamma_1) = L(\gamma_2) \leq L(\gamma'),$$

for all  $\gamma' \in \xi$  such that  $\det(\gamma \gamma') \neq 0$ .

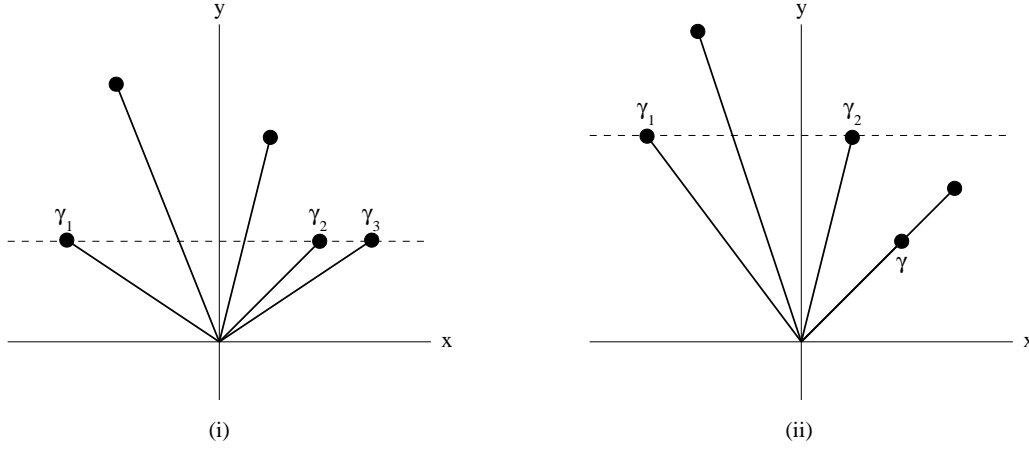


Figure 5: Cases (i) and (ii).

In addition, an element of  $L$ -value 0 could be obtained only after being in one of the cases (i) or (ii). Suppose first that we are not in any of the cases above and that we continue not being in these cases after iterating the algorithm. Now, let us suppose (possibly after renumbering) that  $L(\gamma_i) \leq L(\gamma_r)$ , for all  $1 \leq i \leq r$  and that  $L$  chooses  $\gamma_1$  and  $\gamma_2$ . Apply the algorithm once to obtain  $\xi' = \{\gamma'_1, \dots, \gamma'_{r'}\}$ . Once again, possibly after renumbering, we have  $0 < L(\gamma'_i) \leq L(\gamma'_{r'})$ , for all  $1 \leq i \leq r'$ . Then, since  $\gamma'_{r'} = \gamma_i - \gamma_j$  for some  $i > 2$  and some  $j \in \{1, 2\}$ , we have

$$L(\gamma'_{r'}) = L(\gamma_i) - L(\gamma_j) < L(\gamma_i) \leq L(\gamma_r).$$

Since  $L(\gamma) \in \mathbb{N}$  this situation cannot continue infinitely many times. Therefore, either the resulting semigroup after some iteration of the algorithm is generated by two elements or we arrive at one of the cases (i) or (ii).

So suppose we are in case (i). Let  $k := L(\gamma_1) = L(\gamma_2) = L(\gamma_3)$ . Denote by  $\{\rho_1, \dots, \rho_s\}$  all the elements of  $\xi$  whose  $L$ -value is  $k$ . We can suppose that  $c_x(\rho_1) < c_x(\rho_2) < \dots < c_x(\rho_s)$ , where  $c_x(\rho_i)$  denotes the first coordinate of  $\rho_i$ . Under these assumptions,  $L$  may choose only the couples  $\{\rho_1, \rho_2\}$  or  $\{\rho_{s-1}, \rho_s\}$ . Indeed, let us suppose that  $L$  chooses  $\{\rho_i, \rho_j\}$  different from  $\{\rho_1, \rho_2\}$  and  $\{\rho_{s-1}, \rho_s\}$ . If  $s = 3$ , then

$\{\rho_i, \rho_j\} = \{\rho_1, \rho_3\}$ . This implies that, after applying the algorithm,  $c_x(\rho_2 - \rho_1) > 0$  and  $c_x(\rho_2 - \rho_3) < 0$  and then  $(0, 0) \in \text{Conv}(\rho_2 - \rho_1, \rho_2 - \rho_3) \subset \text{Conv}(\xi') \subset \mathbb{R}^2$ , where  $\xi'$  is the resulting set after applying the algorithm. But according to (B2) of the algorithm, we are supposed to choose only couples such that  $(0, 0) \notin \text{Conv}(\xi')$ , that is, we have a contradiction. If  $s > 3$ , reasoning similarly we have the same conclusion. So let us suppose that  $L$  chooses the couple  $\{\rho_1, \rho_2\}$ . Applying the algorithm one more time will give us  $0 < c_x(\rho_i - \rho_1)$ ,  $0 < c_x(\rho_i - \rho_2)$ ,  $L(\rho_i - \rho_1) = 0$ , and  $L(\rho_i - \rho_2) = 0$  for all  $i > 2$ . Since  $s \geq 3$  we have at least one element in the resulting set whose  $L$ -value is 0, which in this case has the form  $(n, 0)$  with  $n > 0$ . If  $L$  chooses the couple  $\{\rho_{s-1}, \rho_s\}$  then we will obtain an element of the form  $(m, 0)$  with  $m < 0$ .

Now suppose that we are in case (ii). Let  $k := L(\gamma_1) = L(\gamma_2)$ . We denote by  $\{\rho_1, \dots, \rho_s\}$  all the elements of  $\xi$  whose  $L$ -value is  $k$ . Once again, we can suppose that  $c_x(\rho_1) < c_x(\rho_2) < \dots < c_x(\rho_s)$ . Reasoning as before  $L$  chooses  $\gamma$  and could choose only  $\rho_1$  or  $\rho_s$ . Let us suppose that  $L$  chooses  $\rho_1$ . Then  $0 < c_x(\rho_i - \rho_1)$  and  $L(\rho_i - \rho_1) = 0$  for all  $i > 1$  such that  $\det(\rho_i \ \gamma) \neq 0$ . If  $L$  chooses  $\gamma$  and  $\rho_s$  the result is analogous. Since  $s \geq 2$  we have at least one element in the resulting set whose  $L$ -value is 0 which is what we wanted to prove.  $\square$

Now we proceed to prove (2).

**Lemma 3.2.** *If  $\xi = \{(n, 0), \gamma_1, \dots, \gamma_r\}$  is a set of monomial exponents of some toric surface with  $n > 0$ , then the iteration of the algorithm eventually produces a point of the form  $(\lambda, 1)$ , where  $\lambda \in \mathbb{Z}$ .*

*Proof.* Denote by  $\{(n, 0), \rho_1, \dots, \rho_s\}$  the elements of  $\xi$  whose  $L$ -value is 0 and suppose that  $0 < n < c_x(\rho_i)$  for all  $i$ . Then  $L$  first chooses  $(n, 0)$ . Otherwise, since  $L(\rho_i) = 0$  for all  $i$ ,  $L$  is forced to choose some of the  $\rho_i$ , and we would have  $c_x(\rho_i - (n, 0)) < 0$  which contradicts condition (B2). Therefore  $L$  chooses  $(n, 0)$ . The other possible point should be then the one whose first coordinate is the smallest among all the points in the next value of  $L$ .

Denote by  $\{\sigma_1, \dots, \sigma_t\}$  the elements of  $\xi$  whose  $L$ -value is greater than 0 and suppose that  $0 < L(\sigma_1) \leq L(\sigma_2) \leq \dots \leq L(\sigma_t)$  and that  $L$  chooses  $\sigma_1$ . Since  $\mathbb{Z}\xi = \mathbb{Z}^2$ , we have  $\gcd(L(\sigma_1), L(\sigma_2), \dots, L(\sigma_t)) = 1$ . Apply once the algorithm. Then we obtain a new set  $\xi'$  that contains the subset  $\{\sigma_1, \sigma_2 - \sigma_1, \dots, \sigma_t - \sigma_1\}$  (see figure 6).

Since  $\gcd(L(\sigma_1), L(\sigma_2) - L(\sigma_1), \dots, L(\sigma_t) - L(\sigma_1)) = 1$ , we still have that the greatest common divisor of the  $L$ -values of all points in  $\xi'$  is 1. As we did in lemma 2.1, we continue applying the algorithm until we have  $L(\sigma_i) - mL(\sigma_1) < L(\sigma_1)$  for some  $2 \leq i \leq t$  and some  $m \in \mathbb{N}$ . At this moment, we have a new set of monomial exponents with some element whose  $L$ -value is smaller than  $L(\sigma_1)$  and such that

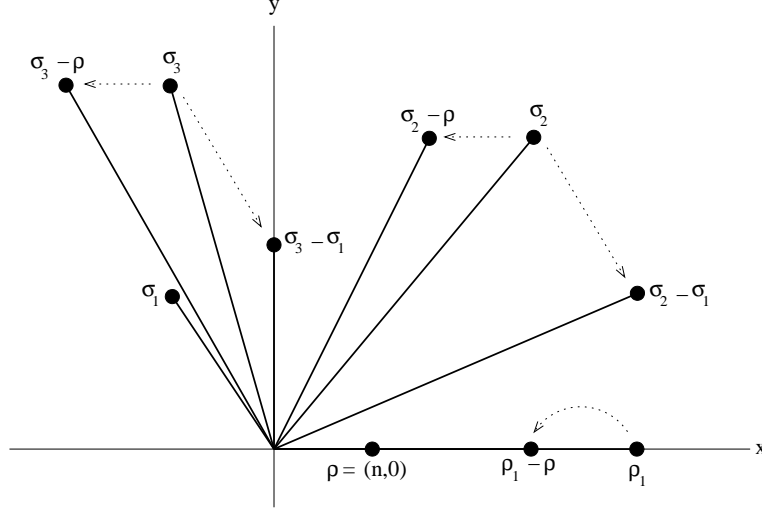


Figure 6: Looking for  $(\lambda, 1)$ .

the greatest common divisor of the  $L$ -values of all its elements is 1, that is, we are in the same situation we began with. Continuing this way, we eventually obtain the desired point. Once we get to some point (or points) whose  $L$ -value is 1, then the one with smallest first coordinate is not generated by the others. As in lemma 2.1, we can assume that this point is  $(0, 1)$ .  $\square$

This lemma allows us to assume that  $(0, 1) \in \xi$ . The next proposition shows that we can obtain some  $(m, 0)$  in the resulting set after applying the algorithm enough times such that  $m < n$ . Since there is always a point  $(\lambda, 1)$  at each step of the algorithm, we will have the same situation but with  $m < n$ . Continuing this way we will eventually obtain the element  $(1, 0)$ .

**Lemma 3.3.** *Let  $\xi = \{(n, 0), (0, 1), \gamma_1, \dots, \gamma_r\}$  be a minimal set of monomial exponents of some toric surface, where  $n > 0$ . Then the iteration of the algorithm eventually produces the point  $(1, 0)$ .*

*Proof.* Suppose that  $(n, 0)$  has the smallest first coordinate among all elements of  $L$ -value 0. We want to find another element whose  $L$ -value is 1 and whose first coordinate is not a multiple of  $n$ . Let  $\xi_n := \xi \cap (n\mathbb{Z} \times \mathbb{Z})$  and  $\xi_0 := \xi \setminus \xi_n$ . Since  $\xi$  is minimal, we may assume that  $(0, 1)$  is the only element of  $L$ -value 1 in  $\xi_n$ . Then  $L$  chooses  $(n, 0)$  and  $(0, 1)$ . If  $\xi'$  is the resulting set after applying the algorithm once, we have  $(\xi_n)' = \xi' \cap (n\mathbb{Z} \times \mathbb{Z})$  and  $(\xi_0)' = \xi' \setminus (\xi_n)'$ . In other words, the elements in  $\xi_n$  only produce elements in  $n\mathbb{Z} \times \mathbb{Z}$  and the elements outside of  $\xi_n$  only produce elements outside of  $n\mathbb{Z} \times \mathbb{Z}$ . Therefore, as long as  $L$  keeps choosing  $(n, 0)$  and  $(0, 1)$ , the effect of the algorithm on  $\xi_n$  is precisely what we saw in proposition

2.3 (see figure 7). In addition, the linear isomorphism we used in that proposition,  $T(x, y) = (x - \lambda y, y)$ , does not change this property if  $\lambda$  is a multiple of  $n$  since, in this case,  $T(\gamma) \in n\mathbb{Z} \times \mathbb{Z}$  if and only if  $\gamma \in n\mathbb{Z} \times \mathbb{Z}$ . All this implies that the effect of the algorithm on  $\xi_0$  is independent of the effect on  $\xi_n$ .

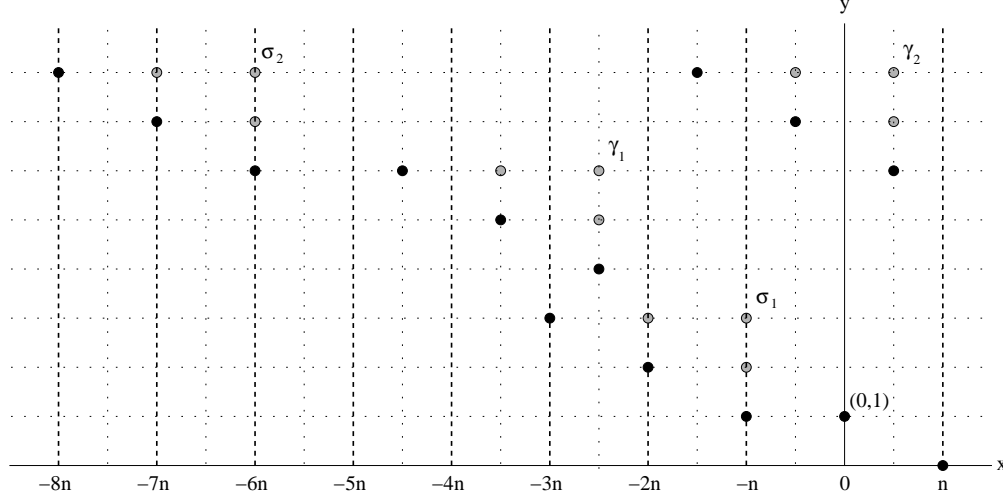


Figure 7:  $\sigma_1, \sigma_2 \in \xi_n$  and  $\gamma_1, \gamma_2 \in \xi_0$ .

Now, since  $\mathbb{Z}\xi = \mathbb{Z}^2$ , there must exist some point  $\gamma \in \xi$  such that  $\gamma \notin n\mathbb{Z} \times \mathbb{Z}$ . Of all these possible elements we consider the one with smallest  $L$ -value and if there are several such points, we take the one whose first coordinate is the smallest. Call this point  $(a, b)$ . We then apply the algorithm  $b - 1$  times. If there is some point in  $\xi_n$  whose  $L$ -value is smaller than  $b$  then we will have to use the isomorphism  $T(x, y) = (x - \lambda y, y)$  after some iteration in order to obtain again the point  $(0, 1)$ . As we said before, this does not change the evolution of the point  $(a, b)$  or its  $L$ -value. So, continuing this way, after these  $b - 1$  times, we obtain another element  $(\lambda, 1)$  different from  $(0, 1)$  and such that  $\lambda$  is not a multiple of  $n$ .

At the next step, there will be some point  $(m, 0)$  different from  $(n, 0)$ . If  $m < n$  we finish. If not, apply the algorithm again to obtain the point  $(m - n, 0)$ . Continuing this way, since  $m$  is not a multiple of  $n$ , we eventually obtain some  $(m', 0)$  with  $m' < n$ . If in this process appears some other point  $(p, 0)$  such that  $0 < p < n$  or  $n < p < m$  the conclusion is the same.  $\square$

**Remark 3.4.** Notice that if  $\xi = \{(n, 0), \gamma_1, \dots, \gamma_r\}$  with  $n < 0$ , then the analogous result (2) for this set can be reduced to the case  $n > 0$  by considering the linear isomorphism  $T(x, y) = (-x, y)$ , since this isomorphism preserves  $L$ .

Putting together the results of this section and the previous one, we finally have,

**Theorem 3.5.** *Let  $\xi \subset \mathbb{Z}^2$  be a set of monomial exponents of some toric surface. Then the iteration of the algorithm following  $L(x, y) = ax + by$ , where  $a, b \in \mathbb{Z}$ , and  $L(\xi) \geq 0$ , eventually produces a semigroup generated by two elements.*

## 4 Counting steps

In this section we are going to prove some results regarding the number of iterations that the algorithm needs to stop in the cases we already solved.

Let  $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$  be a set of monomial exponents of some toric surface and consider  $L(x, y) = y$ . Let

$$\begin{aligned} u_0(\xi) &:= \max\{L(\gamma_i) \mid \gamma_i \in \xi\} \\ u_1(\xi) &:= \min\{L(\gamma_i) \mid \gamma_i \in \xi, \mathbb{Z}(\gamma_{j_0}, \dots, \gamma_{j_s}) = \mathbb{Z}^2 \text{ where } \{\gamma_{j_0}, \dots, \gamma_{j_s}\} \text{ denotes} \\ &\quad \text{the set of all } \gamma_{j_k} \text{ such that } 0 \leq L(\gamma_{j_k}) \leq L(\gamma_i)\} \end{aligned}$$

Suppose that  $L(\gamma_i) > 0$  for all  $i$  and denote by  $\xi_k$  the resulting set after applying the algorithm  $k$  times. Then we have the following two lemmas:

**Lemma 4.1.** *Suppose that after  $u_0(\xi)$  iterations of the algorithm we obtain an element of  $L$ -value 0 for the first time. Then*

- (1)  $0 \leq L(\xi_{u_0(\xi)}) \leq 1$ .
- (2) *There exists some  $\gamma \in \xi_{u_0(\xi)}$  such that  $L(\gamma) = 1$ .*
- (3) *There exist  $\gamma_1, \dots, \gamma_t \in \xi_{u_0(\xi)}$  such that  $L(\gamma_i) = 0$ ,  $t \geq 2$ , and such that  $\gcd(c_x(\gamma_1), \dots, c_x(\gamma_t)) = 1$ , where  $c_x(\gamma_i)$  denotes the first coordinate of  $\gamma_i$ .*

*Proof.* Recall that an element of the form  $(n, 0)$  is produced only after being in one of the cases (i) or (ii) of lemma 3.1. The hypothesis means that only after  $u_0(\xi) - 1$  iterations we arrive to one of these cases. Since after each iteration the value of  $u_0(\cdot)$  decreases at least by one, after  $u_0(\xi) - 1$  iterations all points in the resulting set must have  $L$ -value 1. Another application of the algorithm gives us (1) and (2) for any choice of couples of  $L$ . Let  $\xi_{u_0(\xi)-1} = \{(a_1, 1), (a_2, 1), \dots, (a_r, 1)\}$ , where  $a_1 < a_2 < \dots < a_r$ . Suppose that  $L$  chooses  $(a_1, 1)$  and  $(a_2, 1)$ . Then another application of the algorithm produces

$$\{(a_3 - a_1, 0), \dots, (a_r - a_1, 0)\} \cup \{(a_3 - a_2, 0), \dots, (a_r - a_2, 0)\} \cup \{(a_1, 1), (a_2, 1)\}.$$

Then  $\gcd(a_3 - a_1, \dots, a_r - a_1, a_3 - a_2, \dots, a_r - a_2) = 1$ . Indeed, since  $\mathbb{Z}_{\xi_{u_0(\xi)-1}} = \mathbb{Z}^2$  there exist some  $\lambda_i \in \mathbb{Z}$  such that  $\sum_{i=1}^r \lambda_i(a_i, 1) = (1, 0)$ . Consider the linear isomorphism  $T(x, y) = (x - a_1y, y)$ . Then  $T(\sum_{i=1}^r \lambda_i(a_i, 1)) = T(1, 0) = (1, 0)$ . In particular,  $\sum_{i=2}^r \lambda_i(a_i - a_1) = 1$ , i. e.,  $\gcd(a_2 - a_1, \dots, a_r - a_1) = 1$ . This implies the assertion. If  $L$  chooses  $(a_{r-1}, 1)$  and  $(a_r, 1)$ , we proceed similarly.  $\square$

For the next lemma, rename  $\xi$  as  $\xi_0$ . Now suppose that after  $w < u_0(\xi_0)$  iterations of the algorithm we obtain an element of  $L$ -value 0 for the first time, and denote by  $\xi = \{(n, 0), \gamma_1, \dots, \gamma_r\}$  the resulting set. Let us suppose that  $L$  chooses  $\gamma_0 = (n, 0)$  and  $\gamma_1$ , so, in particular,  $0 = L(\gamma_0) < L(\gamma_1) \leq L(\gamma_j)$ , for all  $\gamma_j \in \xi$  such that  $\det(\gamma_0 \ \gamma_j) \neq 0$ .

**Lemma 4.2.** *Let  $\xi' = \{\gamma'_1, \dots, \gamma'_{r'}\}$  be the resulting set after applying the algorithm once again and suppose that the semigroup  $\mathbb{Z}_{\geq 0}\xi'$  is not generated by two elements. If  $n > 0$  then:*

- (1) *If  $L(\gamma_1) = u_1(\xi)$  then  $L(\gamma_1) = 1$  and  $\xi'$  contains  $(1, 0)$  or at least two elements of  $L$ -value 0 and whose first coordinates are relatively prime. In particular,  $\xi$  contains an element of  $L$ -value 1.*
- (2) *If  $L(\gamma_1) < u_1(\xi)$  then  $u_1(\xi') < u_1(\xi)$ .*
- (3) *If the semigroup  $\mathbb{Z}_{\geq 0}\xi_{u_1(\xi)}$  is not generated by two elements, then  $\xi_{u_1(\xi)}$  contains  $(1, 0)$  or at least two elements of  $L$ -value 0 whose first coordinates are relatively prime, and an element of  $L$ -value 1.*

*Proof.* Let  $\xi^* = \{\gamma_{j_0}, \dots, \gamma_{j_s}\}$  be the elements  $\gamma_j \in \xi$  such that  $0 \leq L(\gamma_j) \leq u_1(\xi)$ . Let  $\gamma \in \xi^*$  be such that  $L(\gamma) = u_1(\xi)$ . Suppose that  $(n, 0) = \gamma_{j_0}$ ,  $\gamma_1 = \gamma_{j_1}$ , and  $\gamma = \gamma_{j_s}$ . By the definition of  $u_1(\xi)$ , we have  $\mathbb{Z}\xi^* = \mathbb{Z}^2$ .

- (1) Suppose that  $L(\gamma_1) = u_1(\xi)$ . Then  $L(\gamma_1) = L(\gamma)$  so  $L(\gamma_{j_k}) = 0$  or  $L(\gamma_{j_k}) = L(\gamma_1)$  for all  $\gamma_{j_k} \in \xi^*$ . Since  $\mathbb{Z}\xi^* = \mathbb{Z}^2$  we have  $\gcd(L(\gamma_{j_0}), \dots, L(\gamma_{j_s})) = 1$ . But then  $L(\gamma_1) > 0$  implies  $L(\gamma_1) = 1$ . If, in addition,  $n = 1$  then we are done. Suppose  $n > 1$ . Then the cardinality of  $\xi^*$  is at least 3. Now proceed as in the previous lemma to find the elements whose first coordinate are relatively prime.
- (2) Suppose now that  $L(\gamma_1) < u_1(\xi)$ . Apply the algorithm once to obtain  $\xi'$ . Consider the subset

$$\xi'^* = \xi'_1 \cup \xi'_2 \cup \{\gamma_0, \gamma_1\},$$

where  $\xi'_1 = \{\gamma_i - \gamma_1 | i \in \{j_2, \dots, j_s\}, L(\gamma_i) > 0\}$  and  $\xi'_2 = \{\gamma_i - \gamma_0 | L(\gamma_i) = 0\}$  (see figure 8). Since  $\xi^* \subset \mathbb{Z}\xi'^*$  then  $\mathbb{Z}^2 = \mathbb{Z}\xi^* \subset \mathbb{Z}\xi'^*$ , that is,  $\mathbb{Z}^2 = \mathbb{Z}\xi'^*$ . Now consider  $l = \max\{L(\gamma) - L(\gamma_1), L(\gamma_1)\}$ . Since  $l \geq L(\gamma_j)$  for all  $\gamma_j \in \xi'^*$  then  $u_1(\xi') \leq l$ . In addition,  $l \leq L(\gamma) = u_1(\xi)$ , so that

$$u_1(\xi') \leq u_1(\xi).$$

Suppose that  $l = u_1(\xi')$ . Then, if  $l = L(\gamma) - L(\gamma_1)$  we have  $u_1(\xi') = l < L(\gamma) = u_1(\xi)$ , since  $L(\gamma_1) > 0$ . If  $l = L(\gamma_1)$  we obtain the same conclusion since, by hypothesis,  $L(\gamma_1) < L(\gamma)$ . So, if  $l = u_1(\xi')$ , for the two possible choices of  $l$ , we have  $u_1(\xi') < u_1(\xi)$ . Otherwise  $u_1(\xi') < l$  and the conclusion follows once again.



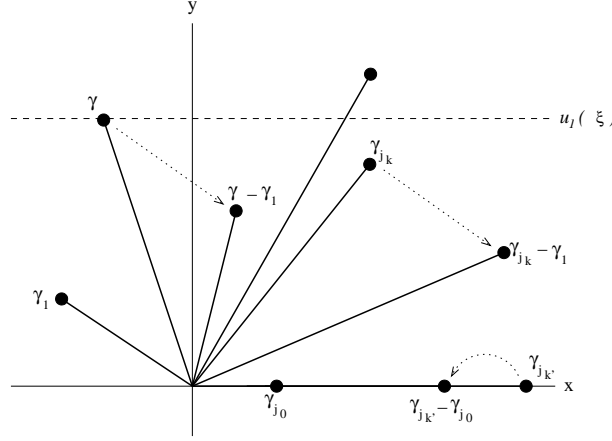


Figure 8:  $L(\gamma_1) < u_1(\xi)$ .

- (3) Since  $1 \leq u_1(\xi)$ , then by (2), after at most  $u_1(\xi) - 1$  iterations, we will obtain  $u_1(\cdot) = 1$ . Then by (1) we conclude. □

**Remark 4.3.** *The analogous result of the previous lemma for  $n < 0$  can be reduced to the case  $n > 0$  by considering the linear isomorphism  $T(x, y) = (-x, y)$ , since this isomorphism preserves  $L$ .*

According to the previous results, after at most  $u_0(\xi)$  iterations, the algorithm will produce, first, an element  $(n, 0)$ , then, some other points of  $L$ -value 0 such that their first coordinates are relatively prime. Of all these points, call  $(N, 0)$  the one with biggest (or smallest if  $n < 0$ ) first coordinate. Our next goal will be to find a bound for  $N$ .

**Lemma 4.4.** *Let  $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$  be a set of monomial exponents of some toric surface such that  $L(\gamma_i) \geq 0$  for all  $i$ . Let  $v_0(\xi) := \max\{|c_x(\gamma_i)| \mid \gamma_i \in \xi\}$ . Let  $\xi_w$  be the resulting set after iterating the algorithm  $w$  times. Then*

$$v_0(\xi_w) \leq 2^w \cdot v_0(\xi).$$

*Proof.* We proceed by induction on  $w$ . For  $w = 1$  it is clear that  $v_0(\xi_1) \leq 2 \cdot v_0(\xi)$  (see figure 9). Suppose that  $v_0(\xi_k) \leq 2^k \cdot v_0(\xi)$ . This means that for all  $\gamma \in \xi_k$  we have  $-2^k \cdot v_0(\xi) \leq c_x(\gamma) \leq 2^k \cdot v_0(\xi)$ , and this is true, in particular, for the two elements chosen by  $L$ . Therefore,  $v_0(\xi_{k+1}) \leq 2^k \cdot v_0(\xi) + 2^k \cdot v_0(\xi) = 2^{k+1} \cdot v_0(\xi)$ , which completes the induction. □

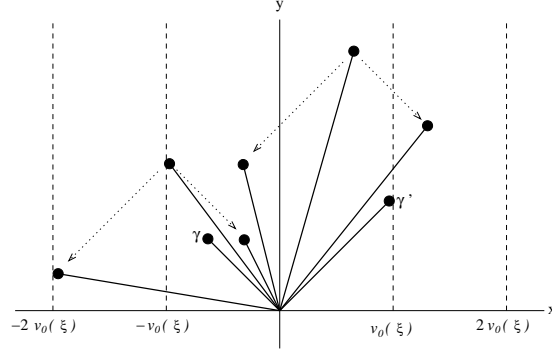


Figure 9:  $v_0(\xi_w) \leq 2^w \cdot v_0(\xi)$ .

**Lemma 4.5.** *Let  $\xi = \{(n_1, 0), \dots, (n_s, 0)\} \cup \{\gamma_1, \dots, \gamma_r\}$  be such that  $0 < L(\gamma_i)$  and  $\gcd(n_1, \dots, n_s) = 1$ . Assume that  $0 < n_1 < n_2 < \dots < n_s$ . If  $n_1 = 1$ , put  $v_1(\xi) := 1$ . If  $n_1 > 1$  let*

$$v_1(\xi) := \min\{n_i \mid \gcd(n_{j_1}, \dots, n_{j_t}) = 1 \text{ where } \{n_{j_1}, \dots, n_{j_t}\} \text{ denotes} \\ \text{the set of all } n_{j_k} \text{ such that } n_{j_k} \leq n_i\}$$

*If  $\xi'$  denotes the resulting set after applying the algorithm once, then  $v_1(\xi') \leq v_1(\xi) - 2$ . Therefore, if  $n_1 > 1$ , after at most  $\lfloor \frac{v_1(\xi)}{2} \rfloor$  iterations we will obtain the element  $(1, 0)$ .*

*Proof.* Since we are looking for the element  $(1, 0)$ , we assume that  $n_1 > 1$ . Suppose that  $n_{i_0} = v_1(\xi)$  where  $2 \leq i_0 \leq s$ . After applying once the algorithm we obtain, in particular, the subset  $\{(n_1, 0), (n_2 - n_1, 0), \dots, (n_{i_0} - n_1, 0)\} \subset \xi'$ . Call  $N = \max\{n_1, n_{i_0} - n_1\}$ . Since  $\gcd(n_1, n_2 - n_1, \dots, n_{i_0} - n_1) = 1$  we have  $v_1(\xi') \leq N$ . If  $N = n_{i_0} - n_1$  then, since  $n_1 \geq 2$  we have  $v_1(\xi') \leq n_{i_0} - n_1 \leq v_1(\xi) - 2$ . Suppose now that  $N = n_1$ . If  $n_{i_0} = n_1 + 1$  then  $n_{i_0} - n_1 = 1$  and  $v(\xi') = 1$  and we are done. Otherwise  $n_{i_0} > n_1 + 1$  which implies  $v_1(\xi') \leq n_1 \leq n_{i_0} - 2$ . This proves the lemma.  $\square$

**Lemma 4.6.** *Let  $\xi = \{(1, 0), (0, 1), \gamma_1, \dots, \gamma_r\}$ . Then after at most  $u_0(\xi)$  iterations, the algorithm stops.*

*Proof.* This is a direct application of the proof of proposition 2.3.  $\square$

**Remark 4.7.** *Analogous results for the two previous lemmas for the cases  $n_s < n_{s-1} < \dots < n_1 < 0$ , or  $(-1, 0)$  instead of  $(1, 0)$ , can be reduced to the previous cases by considering the linear isomorphism  $T(x, y) = (-x, y)$ , since this isomorphism preserves  $L$ .*

Now we are ready to give an estimate of how many iterations are needed for the algorithm to stop. Let  $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$  be a set of monomial exponents of some toric surface. Consider  $L(x, y) = ax + by$  with  $a, b \in \mathbb{Z}$  relatively prime, and such that  $L(\xi) \geq 0$ . Under these conditions, we can suppose, up to linear isomorphism of determinant 1, that  $\xi \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$  and  $L(x, y) = y$ .

**Theorem 4.8.** *Let  $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$  be a set of monomial exponents of some toric surface. Consider  $L(x, y) = y$ . Then after at most*

$$2 \cdot u_0(\xi) + 2^{u_0(\xi)-1} \cdot v_0(\xi)$$

*iterations following  $L$ , the algorithm stops.*

*Proof.* Suppose first that  $L(\gamma_i) > 0$  for all  $i = 1, \dots, r$ . If after exactly  $u_0(\xi)$  iterations we obtain for the first time an element of  $L$ -value 0, say  $(n, 0)$ , then according to lemma 4.1,  $\xi_{u_0(\xi)}$  satisfies  $0 \leq L(\xi_{u_0(\xi)}) \leq 1$ , contains at least two elements of  $L$ -value 0 such that their first coordinates are relatively prime, and at least one element of  $L$ -value 1. In addition,  $v_0(\xi_{u_0(\xi)}) \leq 2^{u_0(\xi)} \cdot v_0(\xi)$  according to lemma 4.4. Therefore, if we do not have it already, by lemma 4.5, after at most  $2^{u_0(\xi)-1} \cdot v_0(\xi)$  iterations we will obtain a set  $\xi'$  that contains  $(1, 0)$  (or  $(-1, 0)$ ). Since  $0 \leq L(\xi_{u_0(\xi)}) \leq 1$ , the set  $\xi'$  also satisfies these inequalities. But now having  $(1, 0)$  (or  $(-1, 0)$ ) implies that the algorithm stops. Summarizing, we needed, at most

$$u_0(\xi) + 2^{u_0(\xi)-1} \cdot v_0(\xi) < 2 \cdot u_0(\xi) + 2^{u_0(\xi)-1} \cdot v_0(\xi)$$

iterations for the algorithm to stop.

Suppose now that after  $w < u_0(\xi)$  iterations the set  $\xi_w$  contains an element  $(n, 0)$ . Rename  $\xi$  as  $\xi_0$  and  $\xi_w$  as  $\xi$ . By lemma 4.2, after  $u_1(\xi)$  iterations, the set  $\xi_{u_1(\xi)}$  contains  $(1, 0)$  (or  $(-1, 0)$  if  $n < 0$ ) or at least two elements of  $L$ -value 0 such that their first coordinates are relatively prime, and at least one element of  $L$ -value 1. In addition,  $v_0(\xi_{u_1(\xi)}) \leq 2^{u_1(\xi)} \cdot v_0(\xi) \leq 2^{u_1(\xi)} \cdot 2^w \cdot v_0(\xi_0)$ , according to lemma 4.4. Therefore, after at most  $2^{u_1(\xi)+w-1} \cdot v_0(\xi_0)$  iterations we will obtain an element  $(1, 0)$  (or  $(-1, 0)$ ), by lemma 4.5. Now we are in the situation of lemma 4.6. Since  $u_0(\xi_k) \leq u_0(\xi_0)$  for any  $k \in \mathbb{N}$ , then after at most  $u_0(\xi_0)$  new iterations the algorithm stops. Summarizing, we needed, at most

$$w + u_1(\xi) + 2^{u_1(\xi)+w-1} \cdot v_0(\xi_0) + u_0(\xi) \leq 2 \cdot u_0(\xi_0) + 2^{u_0(\xi_0)-1} \cdot v_0(\xi_0)$$

iterations for the algorithm to stop. The inequality follows since  $u_1(\xi) \leq u_0(\xi) \leq u_0(\xi_0) - w$ .

Finally, if  $\xi$  already contains some element of  $L$ -value 0 then we are in the same situation as in the previous paragraph without doing the first  $w$  iterations. Therefore the result follows similarly. This proves the theorem.  $\square$

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